P03

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ON THE USE OF BREGMAN DISTANCES FOR THE SOLUTION OF INVERSE RADIATIVE TRANSFER PROBLEMS IN ONE-DIMENSIONAL PARTICIPATING MEDIA

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Abstract – Tikhonov's regularization approach has been applied in several inverse problems converting an *ill-posed* problem into a *well-posed* one. This work proposes the use of Bregman distances based on the *q-discrepancy* functional as regularization terms. Here the inverse problem is formulated implicitly as an optimization problem in which we seek to minimize the cost functional of squared residues between calculated and measured quantities. Such approach is then applied for the estimation of the single scattering albedo, optical thickness and inner boundary diffuse reflectivities in a one-dimensional participating medium.

1. INTRODUCTION

The solution of inverse radiative transfer problems has several relevant applications, not only in engineering [1,14,20,33], but also in many other areas such as astrophysics [16,25], physical oceanography (optical hydrology) [15,17,28,32], remote sensing [9,22], and atmosphere / hydrosphere optics [11,31].

When formulated implicitly inverse radiative problems usually require the minimization of a cost functional related to the squared residues between an observable quantity and the calculated value for such quantity.

As inverse problems are usually *ill-posed* they are affected by the noise which is always present in the experimental data. An effective strategy to solve such problems is to replace the original inverse problem of interest by another one that is close to the former but is less affected by the experimental data noise. Such approach was developed by Tikhonov [29]. In Tikhonov's approach a regularized functional is developed, in which an extra term, that may include some prior information related to the unknowns to be estimated, is added to the original objective function to be minimized. Several works are related to the use, analysis and proposition of Tikhonov's regularization terms [5,18,23,24].

In the present work we build a family of regularizing terms using Bregman distances constructed using moments of a *q-discrepancy* functional [2,21]. They are very simple to implement computationally and seem to yield a robust algorithm which provides reasonable estimates even when noisy data is used. Such approach is applied to the solution of inverse radiative transfer problems. In these problems we are interested in the estimation of the single scattering albedo, optical thickness and inner boundary diffuse reflectivities in a one-dimensional participating medium. Only experimental data acquired by external detectors is considered. As experimental data we use synthetic values of the intensity of the exit radiation as a function of the polar angle.

2. DIRECT AND INVERSE RADIATIVE TRANSFER PROBLEM

2.1 Mathematical formulation of the direct and inverse problems

The phenomena associated with the interaction of radiation with a participating medium, as schematically represented in Figure 1, i.e., an absorbing emitting and scattering media, are usually mathematically modeled using the dimensionless linear version of the Boltzmann equation, also referred to as transport equation, which is written as [19]

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = \frac{\omega}{2}\int_{-1}^{1} I(\tau,\mu')\, d\mu' + S(\tau,\mu) \tag{1}$$

subject to the following boundary conditions

$$I(0,\mu) = f_1(\mu) + 2\rho_1 \int_0^1 I(0,-\mu')\, \mu'\, d\mu', \quad \mu > 0 \tag{2}$$

and

$$I(\tau_0,-\mu) = f_2(\mu) + 2\rho_2 \int_0^1 I(\tau_0,\mu')\, \mu'\, d\mu', \quad \mu > 0 \tag{3}$$

where I is the dimensionless radiation intensity, τ is the optical variable, τ_0 is the optical thickness of the medium, μ is the direction cosine of the radiation beam with the positive τ axis, that is, the cosine of the polar angle ($\mu = \cos\theta$), ω is the single scattering albedo, ρ_1 and ρ_2 are the diffuse reflectivities at the inner side of the boundary surfaces at $\tau = 0$ and $\tau = \tau_0$, S represents an internal radiation source, and f_1 and f_2 are the intensities of the incident radiation at $\tau = 0$ and $\tau = \tau_0$.

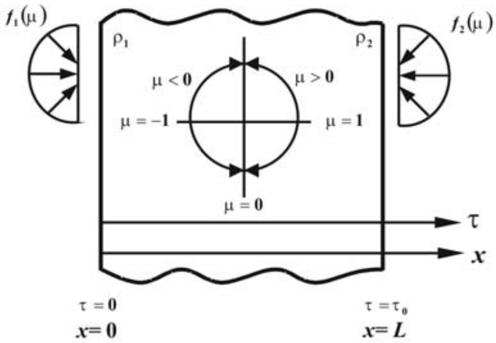

Figure 1. Schematical representation of a one-dimensional homogeneous participating medium, with inner diffusely reflecting boundary surfaces, subjected to isotropic external radiation.

When the boundary conditions and the material properties are known, the direct problem (1)-(3) can be solved providing the values of the radiation intensity for every point in the spatial and angular domains. This is the so called direct problem. Here we use Chandrasekhar's discrete ordinates method [6] to solve the direct radiative transfer problem.

In the inverse problem considered in the present work the optical thickness τ_0, the single scattering albedo ω and the diffuse reflectivities ρ_1 and ρ_2 are the unknowns to be determined [27].

As these properties cannot be measured directly, experimental data on the intensity of the exit radiation at $\tau = 0$ and $\tau = \tau_0$ is measured at different polar angles, and using this information we try to solve the inverse problem.

Therefore, by using the measured data Y_i, $i = 1,2,...,N_d$, where N_d is the total number of experimental data available, we want to determine the elements of the vector of unknowns $\vec{P}$ defined as

$$\vec{P} = \{\tau_0, \omega, \rho_1, \rho_2\}^T \tag{4}$$

where the superscript T indicates transpose.

As the number of experimental data is usually larger than the number of unknowns, we formulate the inverse problem implicitly as a finite dimensional optimization problem, in which we seek to minimize the Tikhonov's regularization functional

$$T(\vec{P}) = \sum_{i=1}^{N_d} \left[I_{calc_i}(\vec{P}) - Y_i \right]^2 + \alpha\, R(\vec{P}, \vec{P}^R) \tag{5}$$

where I_{calc_i} represents the calculated values for the exit radiation intensity obtained using an estimate for the vector of unknowns $\vec{P}$, α is the regularization parameter, R is the regularization term, and $\vec{P}^R$ is a vector with reference values (prior information) for the unknowns.

2.2 General regularization term for the Tikhonov functional

In the present work we use a family of regularization terms R for Tikhonov's functional shown in eqn.(5). These regularization terms are Bregman distances [3] constructed with the moments of the *q-discrepancy* [2,21], and the *q-discrepancy* is a particular case of Csiszár's measure [7].

Bregman distances

In this section we provide a brief description of Bregman distances. Most of the material presented here is collected from the references [4,8,12].

Let $\eta : \Delta \subseteq \Re^n \to \Re$ be a differentiable and strictly convex function defined on a closed, convex set Δ. The Bregman distance associated with $\eta(.)$ is defined for $p, p^R \in \Delta$ to be

$$B_\eta(p, p^R) = \eta(p) - \eta(p^R) - \left\langle \nabla\eta(p^R), (p - p^R) \right\rangle \tag{6}$$

where the function $\eta(.)$ is called the *Bregman function*. It can be shown that, in general, every Bregman distance is nonnegative

$$B_\eta(p, p^R) \geq 0 \tag{7}$$

and is equal to zero if its two arguments are equal,

$$B_\eta(p, p^R) = 0 \Leftrightarrow p = p^R \tag{8}$$

The conditions (7) and (8) are absolutely essential, and convexity is highly desirable for mathematical convenience. Bregman distances [3] can be interpreted as a measure of convexity of $\eta(.)$. A graphical representation of $\eta(.)$ as well as the measure of convexity of $\eta(.)$ is shown in Figure 2. In the one-dimensional case, the Bregman distance is easy to visualize: drawing a tangent line to the graph of η at the point p^R, the Bregman distance $B_\eta(p, p^R)$ is seen as the vertical distance between this line and the point $\left(p, \eta(p, p^R)\right)$ [4,8,13].

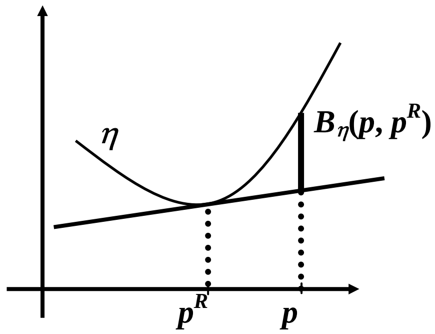

Figure 2. A graphical representation of Bregman distances (redraw from [4]).

Due to the convexity of Bregman distances, they may be very convenient to construct the regularization terms for Tikhonov's functional.

Moments of the q-discrepancy

In the present work the Bregman distances $D_{m,q}\left(\vec{P}, \vec{P}^R\right)$ were built starting from the Csiszár's measure [10], which we call *q-discrepancy*. The moment of *m*th-other of the *q-discrepancy* is defined as

$$\eta_{m,q}\left(\vec{P}\right) = \sum_{i=1}^{N_u} p_i^m \frac{p_i^q - 1}{q} \tag{9}$$

where N_u is the total number of unknowns in the inverse problem we want to solve. Here $N_u = 4$, as shown in eqn.(4). This functional represents the deviation of an expected value of each property of the medium p_i raised to the power q. The parameter q is a real positive number.

The functional (9) for the particular case when $q \to 0$ corresponds, therefore, to the entropy functional

$$\eta_{m,0}\left(\vec{P}\right) = \sum_{i=1}^{N_u} p_i^m \ln p_i \tag{10}$$

Equation (10) is similar to the Boltzmann-Gibbs-Shannon formulation for entropy with the value of $q \to 0$ in our formulation, or similar to the expression proposed by Tsallis [30] when the limit $q \to 1$ is taken in his formulation [10,26].

Regularization terms and Tikhonov functional

Introducing eqn.(9) into eqn.(6), results in

$$B_{\eta_{m,q}}(\vec{P}, \vec{P}^R) = D_{m,q}(\vec{P}, \vec{P}^R) = \sum_{i=1}^{N_u} \frac{p_i^m\left(p_i^q - 1\right)}{q} - \sum_{i=1}^{N_u} \frac{p_i^{R^m}\left(p_i^{R^q} - 1\right)}{q} - \sum_{i=1}^{N_u}\left(p_i - p_i^R\right)p_i^{R^{m-1}}\left[\frac{(m+q)\,p_i^{R^q} - m}{q}\right] \tag{11}$$

which for the particular case $q \to 0$ yields

$$B_{\eta_{m,0}}(\vec{P}, \vec{P}^R) = D_{m,0}(\vec{P}, \vec{P}^R) = \sum_{i=1}^{N_u} p_i^m \ln p_i - \sum_{i=1}^{N_i} p_i^{R^m} \ln p_i^R - \sum_{i=1}^{N_u}\left(p_i - p_i^R\right)p_i^{R^{m-1}}\left(1 + m \ln p_i^R\right) \tag{12}$$

Using the Bregman distances as the regularization terms in Tikhonov's functional, we get from eqns (5) and (11),

$$T\left(\vec{P}\right)=\sum_{i=1}^{N_d}\left[I_{calc_i}\left(\vec{P}\right)-Y_i\right]^2+\alpha\, D_{m,q}\left(\vec{P},\vec{P}^R\right)=\vec{F}^T\vec{F}+\alpha\, D_{m,q}\left(\vec{P},\vec{P}^R\right) \tag{13}$$

where the elements of the vector of residues $\vec{F}$ are given by

$$F_i = I_{calc_i}\left(\vec{P}\right)-Y_i, \qquad i=1,2,...,N_d \tag{14}$$

3. SOLUTION OF THE INVERSE PROBLEM

We are looking for the vector $\vec{P}$, which minimizes the Tikhonov functional given by eqn.(13). For that purpose we first write the critical point equation

$$\frac{\partial T(\vec{P})}{\partial \vec{P}}=0 \therefore \frac{\partial T(\vec{P})}{\partial p_j}=\sum_{i=1}^{N_d}\left[I_{calc_i}(\vec{P})-Y_i\right]\frac{\partial I_{calc_i}(\vec{P})}{\partial p_j}+\alpha\frac{\partial D_{m,q}(\vec{P},\vec{P}^R)}{\partial p_j}=0, \quad j=1,2,...,N_u \tag{15}$$

The system of nonlinear eqns (15) is written in compact form as

$$J^T\vec{F}+\alpha\nabla D_{m,q}=0 \tag{16}$$

where the elements of the Jacobian matrix J are given by

$$J_{st}=\frac{\partial I_{calc_s}}{\partial p_t}, \qquad s=1,2,...,N_d, \qquad t=1,2,...,N_u \tag{17}$$

and

$$\nabla D_{m,q}=\left\{\frac{\partial D_{m,q}}{\partial p_1},\frac{\partial D_{m,q}}{\partial p_2},\ldots,\frac{\partial D_{m,q}}{\partial p_{N_u}}\right\} \tag{18}$$

The elements of the Jacobian matrix J, eqn.(17), were calculated numerically using a central difference approximation.

Writing the Taylor's expansions

$$\vec{F}\left(\vec{P}^{n+1}\right)=\vec{F}\left(\vec{P}^n\right)+J^n\Delta\vec{P}^n \tag{19}$$

and

$$\nabla D_{m,q}\left(\vec{P}^{n+1}\right)=\nabla D_{m,q}\left(\vec{P}^n\right)+J_D^n\Delta\vec{P}^n \tag{20}$$

where n will be the iteration index for the iterative procedure that will be constructed for the estimation of the vector of unknowns $\vec{P}$, with

$$\vec{P}^{n+1}=\vec{P}^n+\Delta\vec{P}^n \tag{21}$$

and the elements of the Jacobian matrix J_D are given by

$$J_{D_{u,v}}=\frac{\partial}{\partial p_u}\left(\frac{\partial D_{m,q}}{\partial p_v}\right), \qquad u=1,2,\ldots,N_u, \quad v=1,2,\ldots,N_u \tag{22}$$

such that

$$J_{D\,u,v}=\begin{cases}\dfrac{\partial^2 D_{m,q}}{\partial p_u^2} & \text{if} \quad u=v\\ 0 & \text{if} \quad u\neq v\end{cases}, \quad u=1,2,\ldots,N_u, \quad v=1,2,\ldots,N_u \tag{23}$$

and introducing eqns (19) and (20) into eqn.(16) results in

$$\left(J^{n^T}J^n+\alpha\, J_D^n\right)\Delta\vec{P}^n=-\left(J^{n^T}\vec{F}^n+\alpha\,\nabla D_{m,q}^n\right) \tag{24}$$

In Table 1 are presented the expressions for $\dfrac{\partial D_{m,q}}{\partial p_j}$ and $\dfrac{\partial^2 D_{m,q}}{\partial p_j^2}$ calculated from eqn.(11) when $q>0$, and from eqn.(12) when $q\to 0$.

Starting with an initial guess $\vec{P}^0$, new estimates for the unknowns are obtained from eqn.(21) with the corrections $\Delta\vec{P}^n$ being obtained from the solution of the system of linear algebraic eqns (24).

The iterative procedure is interrupted when a stopping criterion is satisfied, such as

$$\left|\frac{\Delta p_i^n}{p_i^n}\right|<\varepsilon, \qquad i=1,2,\ldots,N_u \tag{25}$$

where ε is a tolerance, say 10^{-5}.

Table 1: First and second derivatives of the Bregman distances based on the moments of the *q*-discrepancy for $q > 0$ and $q \to 0$.

$q > 0$	$q \to 0$
$\frac{\partial D_{m,q}}{\partial p_j} = p_j^{m-1}\left[\frac{(m+q)\,p_j^q - m}{q}\right] - p_j^{R^{m-2}}\left[\frac{(m+q)\,p_j^{R^q} - m}{q}\right]$	$\frac{\partial D_{m,0}}{\partial p_j} = p_j^{m-1}\left(1+m\ln p_j\right) - p_j^{R^{m-1}}\left(1+\ln p_j^R\right)$
$\frac{\partial^2 D_{m,q}}{\partial p_j^2} = p_j^{m-2}\left[\frac{(m+q)(m+q-1)\,p_j^q - m(m-1)}{q}\right]$	$\frac{\partial^2 D_{m,0}}{\partial p_j^2} = p_j^{m-2}\left[m+(m-1)(1+m\ln p_j)\right]$

4. RESULTS AND DISCUSSION

As real experimental data was not available we generated synthetic experimental data by adding noise to the calculated values of the radiation intensity,

$$Y_i = I_{calc_i}\left(\vec{P}_{exact}\right) + \sigma\, e_i\,, \qquad i = 1,2,\ldots,N_d \tag{26}$$

where σ is the standard deviation of the experimental errors and e_i represent random numbers in range [−1,1].

In this work we have considered a Gauss-Legendre quadrature to replace the integral term in eqn.(1), with $N = 20$ collocation points. The synthetic experimental data was generated for the same grid points considered for the angular domain discretization, and therefore $N_d = N = 20$, being half of the experimental data acquired at $\tau = 0$ and half at $\tau = \tau_0$.

In order to demonstrate the efficacy of the use of the Bregman distances as the regularization terms in Tikhonov's functional, we have chosen two sets of exact values and initial guesses for which the algorithm without regularization did not converge. The exact values and the initial estimates for the radiative properties are shown in Table 2.

Table 2: Exact values and initial estimates for the test cases.

Case	Exact parameters $\vec{P}_{exact} = \{\tau_0, \omega, \rho_1, \rho_2\}^T$	Initial guess $\vec{P}^0 = \{\tau_0^0, \omega^0, \rho_1^0, \rho_2^0\}$
1	{1.0, 0.5, 0.1, 0.95}	{5.0, 0.95, 0.95, 0.1}
2	{0.3, 0.5, 0.1, 0.95}	{1.5, 0.95, 0.95, 0.1}

In Tables 3a-c are presented the estimated values for Test Case 1 at some specific iterations using, respectively, $\sigma = 0.0$, $\sigma = 0.025$ and $\sigma = 0.05$, corresponding to errors in the experimental data up to 0 %, 14% and 35 %. These estimates were obtained with *q* = 1.5, *m* = 1.0, and the regularization parameter $\alpha = 0.01$.

In Tables 4a-c are presented the estimated values for Test Case 2 at some specific iterations using, respectively, $\sigma = 0.0$, $\sigma = 0.025$ and $\sigma = 0.05$, corresponding to errors in the experimental data up to 0 %, 9 % and 16 %. The estimates were obtained with regularization using the parameters *q* = 1.5, *m* = 1.0, and the regularization parameter $\alpha = 0.01$.

From Tables 3 and 4 we observe that the algorithm converged even when poor initial guesses are used. We observe also that for the runs with noisy data poor estimates may be obtained for parameter ρ_1. This is expected since for all cases run in this paper we have considered a difficult situation in which $f_1 = 1.0$ and $f_2 = 0.0$ in eqns (2) and (3). A sensitivity analysis has shown the lowest sensitivity of the experimental data (intensity of the exit radiation) with respect to the parameter ρ_1.

We have then varied the value of the parameters *q* and *m* in the ranges $0.0 < q \le 2.5$ and $0 \le m \le 3$. In Tables 5a-c and 6a-c we show for Test Cases 1 and 2, respectively, the pairs (*q*, *m*) for which convergence was achieved, indicated by C, and those for which convergence was not achieved, indicated by NC. For these runs we have kept the same value for the regularization parameter $\alpha = 0.01$.

From Tables 5 and 6 we observe that convergence is not achieved for some values of the parameters (*q*, *m*). There is a trend of non-convergence when higher values are used for both parameters *q* and *m*. We have performed an analysis of the Bregman distances, given by eqn.(11), constructed with the moments of the *q*-discrepancy, given by eqn.(9), and it seems that convexity is not achieved for such high values of the *q* and *m* parameters. This subject must be further investigated.

Table 3: Estimated values for the unknowns at some specific iterations with regularization, q = 1.5, m = 1.0, $\alpha = 0.01$. Test Case 1. $\vec{P}_{exact} = \vec{P}^{R} = \{1.0, 0.5, 0.1, 0.95\}$.

(3a) $\sigma = 0.0$ (noiseless data)

Iteration	τ_0	ω	ρ_1	ρ_2	Objective Function (eqn.(5))
0	5.0	0.95	0.95	0.1	0.22677
1	1.9574	8.6165e-01	7.8050e-01	7.4832e-01	6.0228e-01
2	1.0866	7.7755e-01	6.1642e-01	9.9325e-01	1.0725e-01
3	9.2517e-01	5.4272e-01	2.5327e-01	9.6960e-01	9.6624e-03
4	9.9953e-01	5.0508e-01	1.2281e-01	9.5286e-01	7.1872e-04
5	9.9990e-01	5.0017e-01	1.0088e-01	9.5005e-01	1.3081e-05
⋮	⋮	⋮	⋮	⋮	⋮
10	1.0000	5.0000e-01	9.9999e-02	9.4999e-01	-1.0121e-19
⋮	⋮	⋮	⋮	⋮	⋮
15	1.0000	5.0000e-01	9.9999e-02	9.4999e-01	-7.0022e-19
⋮	⋮	⋮	⋮	⋮	⋮
18	1.0000	5.0000e-01	9.9999e-02	9.4999e-01	2.2138e-19
⋮	⋮	⋮	⋮	⋮	⋮

(3b) σ = 0.025 (up to 14 % error in experimental data)

Iteration	τ_0	ω	ρ_1	ρ_2	Objective Function (eqn.(5))
0	5.0	0.95	0.95	0.1	0.22677
1	1.9580	8.6589e-01	7.7792e-01	8.3664e-01	6.0794e-01
2	1.0905	8.1383e-01	6.6645e-01	1.0042	6.6621e-02
3	9.0516e-01	5.8671e-01	3.1475e-01	9.6875e-01	1.6765e-02
4	9.6898e-01	5.2624e-01	1.7396e-01	9.4764e-01	5.7274e-03
5	9.7147e-01	5.2266e-01	1.5504e-01	9.4429e-01	3.8207e-03
⋮	⋮	⋮	⋮	⋮	⋮
10	9.7141e-01	5.3373e-01	1.5502e-01	9.4431e-01	3.8063e-03
⋮	⋮	⋮	⋮	⋮	⋮
15	9.7141e-01	5.3373e-01	1.5502e-01	9.4431e-01	3.8063e-03
⋮	⋮	⋮	⋮	⋮	⋮
18	9.7141e-01	5.3373 e-01	1.5502e-01	9.4431e-01	3.8063e-03
⋮	⋮	⋮	⋮	⋮	⋮

(3c) σ = 0.05 (up to 35 % error in experimental data)

Iteration	τ_0	ω	ρ_1	ρ_2	Objective Function (eqn.(5))
0	5.0	0.95	0.95	0.1	0.22677
1	1.9643	8.7257e-01	7.9183e-01	1.0503	5.9399e-01
2	1.0625	8.0581e-01	6.8539e-01	9.5979e-01	4.3788e-02
3	8.3460e-01	5.7396e-01	3.6579e-01	9.3023e-01	2.0714e-02
4	9.1976e-01	4.8215e-01	1.6158e-01	9.2986e-01	1.4420e-02
5	9.2557e-01	4.6374e-01	8.6082e-02	9.3550e-01	1.2279e-02
⋮	⋮	⋮	⋮	⋮	⋮
10	9.3100e-01	4.5439e-01	4.9884e-02	9.3543e-01	1.2036e-02
⋮	⋮	⋮	⋮	⋮	⋮
15	9.3100e-01	4.5439e-01	4.9884e-02	9.3543e-01	1.2036e-02
⋮	⋮	⋮	⋮	⋮	⋮
18	9.3100e-01	4.5439e-01	4.9884e-02	9.3543e-01	1.2036e-02
⋮	⋮	⋮	⋮	⋮	⋮

Table 4: Estimated values for the unknowns at some specific iterations with regularization, q = 1.5, m = 1.0, $\alpha = 0.01$. Test Case 2. $\vec{P}_{exact} = \vec{P}^{R} = \{0.3, 0.5, 0.1, 0.95\}$.

(4a) $\sigma = 0.0$ (noiseless data)

Iteration	τ_0	ω	ρ_1	ρ_2	Objective Function (eqn.(5))
0	1.5	0.95	0.95	0.1	6.12294
1	7.6193e-01	8.3812e-01	4.2532e-01	4.2333e-01	3.3177
2	-9.3717e-02	6.2962e-01	6.4653e-02	1.2013	9.9722e-01
3	1.4720e-01	5.0703e-01	-4.5323e-01	8.8438e-01	1.1987e-01
4	2.6073e-01	5.8709e-01	5.2585e-01	9.8524e-01	1.2016e-01
5	3.0671e-01	4.9577e-01	1.9590	9.6072e-01	1.4125e-02
⋮	⋮	⋮	⋮	⋮	⋮
10	2.9999e-01	5.0000e-01	9.9999e-02	9.4999e-01	6.1291e-18
⋮	⋮	⋮	⋮	⋮	⋮
15	2.9999e-01	5.0000e-01	9.9999e-02	9.4999e-01	4.4608e-18
⋮	⋮	⋮	⋮	⋮	⋮
18	2.9999e-01	5.0000e-01	9.9999e-02	9.4999e-01	3.5614e-18
⋮	⋮	⋮	⋮	⋮	⋮

(4b) σ = 0.025 (up to 9 % error in experimental data)

Iteration	τ_0	ω	ρ_1	ρ_2	Objective Function (eqn.(5))
0	1.5	0.95	0.95	0.1	6.12294
1	7.8553e-01	8.4381e-01	4.1771e-01	4.0183e-01	3.2368
2	9.0102e-02	6.7891e-01	5.7351e-02	1.1495	9.7026e-01
3	3.4418e-01	9.1890e-01	8.3298e-01	9.4345e-01	6.3238e-01
4	-1.5991e-01	5.9698e-01	4.1595e-01	8.5763e-01	1.0995e-01
5	1.3588e-01	5.0577e-01	3.8936e-01	8.2903e-01	5.1127e-01
⋮	⋮	⋮	⋮	⋮	⋮
10	2.8710e-01	5.5444e-01	2.0660e-01	9.3643e-01	5.2828e-03
⋮	⋮	⋮	⋮	⋮	⋮
15	2.8713e-01	5.5444e-01	2.0651e-01	9.3642e-01	5.2828e-03
⋮	⋮	⋮	⋮	⋮	⋮
18	2.8713e-01	5.5444e-01	2.0651e-01	9.3642e-01	5.2828e-03
⋮	⋮	⋮	⋮	⋮	⋮

(4c) σ = 0.05 (up to 16 % error in experimental data)

Iteration	τ_0	ω	ρ_1	ρ_2	Objective Function (eqn.(5))
0	1.5	0.95	0.95	0.1	6.12294
1	8.0789e-01	8.4726e-01	4.1038e-01	4.7814e-01	3.3876
2	1.0567e-01	7.2322e-01	2.1043e-01	1.2041	8.4015e-01
3	3.5646e-01	8.1281e-01	4.6527e-01	9.6099e-01	5.4613e-01
4	1.9296e-01	6.0576e-01	2.7711e-01	9.5253e-01	6.2781e-02
5	2.5811e-01	5.8661e-01	3.5315e-01	9.5869e-01	7.2982e-02
⋮	⋮	⋮	⋮	⋮	⋮
10	2.5479e-01	5.7496e-01	3.4531e-01	9.5889e-01	2.0335e-02
⋮	⋮	⋮	⋮	⋮	⋮
15	2.5479e-01	5.7496e-01	3.4531e-01	9.5889e-01	2.0335e-02
⋮	⋮	⋮	⋮	⋮	⋮
18	2.5479e-01	5.7496e-01	3.4531e-01	9.5889e-01	2.0335e-02
⋮	⋮	⋮	⋮	⋮	⋮
20	2.5479e-01	5.7496e-01	3.4531e-01	9.5889e-01	2.0335e-02
⋮	⋮	⋮	⋮	⋮	⋮

Table 5: Regions (q, m) of convergence (C) and non-convergence (NC) for the algorithm with regularization. $\alpha = 0.01$. Test Case 1. $\vec{P}_{exact} = \vec{P}^{R}$.

(5a) $\sigma = 0.0$

m \ q	0.0	0.5	1.0	1.5	2.0	2.5
0.0	NC	NC	C	C	C	C
1.0	C	C	C	C	C	C
2.0	NC	NC	NC	NC	NC	NC
3.0	NC	NC	NC	NC	NC	NC

(5b) $\sigma = 0.025$

m \ q	0.0	0.5	1.0	1.5	2.0	2.5
0.0	NC	NC	NC	C	C	NC
1.0	C	C	C	C	C	C
2.0	NC	NC	NC	NC	NC	NC
3.0	NC	NC	NC	NC	NC	NC

(5c) $\sigma = 0.05$

m \ q	0.0	0.5	1.0	1.5	2.0	2.5
0.0	NC	NC	NC	C	C	NC
1.0	C	C	C	C	C	C
2.0	NC	NC	NC	NC	NC	NC
3.0	NC	NC	NC	NC	NC	NC

Table 6: Regions (q, m) of convergence (C) and non-convergence (NC) for the algorithm with regularization. $\alpha = 0.01$. Test Case 2. $\vec{P}_{exact} = \vec{P}^{R}$.

(6a) $\sigma = 0.0$

m \ q	0.0	0.5	1.0	1.5	2.0	2.5
0.0	NC	NC	C	C	C	C
1.0	C	C	C	C	C	C
2.0	C	C	NC	NC	NC	NC
3.0	NC	NC	NC	C	C	C

(6b) $\sigma = 0.025$

m \ q	0.0	0.5	1.0	1.5	2.0	2.5
0.0	NC	NC	NC	C	C	NC
1.0	C	NC	NC	C	NC	NC
2.0	NC	NC	NC	NC	NC	NC
3.0	NC	NC	NC	NC	NC	NC

(6c) $\sigma = 0.05$

m \ q	0.0	0.5	1.0	1.5	2.0	2.5
0.0	NC	NC	NC	C	C	NC
1.0	C	NC	NC	C	NC	NC
2.0	NC	NC	NC	NC	NC	NC
3.0	NC	NC	NC	NC	NC	NC

5. CONCLUSIONS

In the present work we presented the efficacy of using Bregman distances constructed with the moments of *q-discrepancy* as regularization terms in Tikhonov's functional for the solution of inverse radiative transfer problems. We found that in some ranges of the control parameters convexity is not observed. This subject will be further investigated in future works, and weighting factors will be implemented as an attempt to keep the convexity of the regularization terms.

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